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Fluctuation-Induced Couplings between Defect Lines or Particle Chains

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One-dimensional structures such as defect lines or chains of dipolar particles are generally subject to strong Landau–Peierls thermal fluctuations. Coupling between these fluctuations in parallel lines may lead to an attractive force, analogous to the London force, or to a repulsive force of entropic origin. We analyze these forces for chains of electric dipoles and for flux lines in isotropic superconductors. In the first case the force is attractive, and can significantly change the Hamaker constant, which governs the attraction between colloidal particles. In the second case, over much of the magnetic field–temperature phase diagram the force is repulsive, and dominates over the direct repulsive interaction between flux lines.

KEY WORDS: Electrorheology; flux lattice; van der Waals forces.

1. INTRODUCTION

In many contexts in condensed matter physics, one-dimensional structures such as defect lines or chains of particles play a significant role. The most famous such defects are certainly Abrikosov flux lines in superconductors, but vortex lines in liquid helium and in liquid crystal systems, and defect lines in magnetic systems, behave in closely analogous ways.[1–3] In all of these cases, the lines represent topological defects in an underlying order parameter, and the dynamics of the lines can be understood in terms of the dynamics of that order parameter.[4]

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Another type of one-dimensional structure arises in colloidal physics. Chains of particles play an important role in ferrofluids and in electrorheological fluids, where respectively magnetic and electric dipole interactions between colloidal particles lead them to form roughly linear chains parallel to an applied field.[5,6]

These chains and lines, being one-dimensional structures, will experience strong Landau–Peierls fluctuations.[7] Suppose that the deviation of the line or chain from its mean position is written as $\mathbf{a}(z)$, where z is the coordinate along the length of the line. Naively, we might expect the energy cost associated with this deviation to be proportional to $(\partial_z \mathbf{a})^2$. It follows that the thermal expectation value $\langle \mathbf{a}^2(z) \rangle$ is given by a momentum space integral,

$$\langle \mathbf{a}^2(z) \rangle \propto \int \frac{dk}{k^2} \tag{1.1}$$

which diverges with the size of the system.

Recently this fact has motivated very active theoretical and experimental study of the behavior of Abrikosov vortex lines in high-temperature superconductors, where these fluctuations may alter the macroscopic behavior of the mixed phase of the superconductors.[8,9] It seems that under the right circumstances, a set of vortex lines will form a vortex liquid, possibly entangled, instead of the traditional Abrikosov vortex lattice. Larkin has argued that the vortex lattice is unstable with respect to quenched disorder in a material; thus, another possibility is that a phase with slow dynamics, which may or may not be a thermodynamic "vortex glass," replaces the Abrikosov solid.[10,11]

In this study we will address a different consequence of the Landau–Peierls fluctuations. In statistical mechanics, we are accustomed to the idea that thermal or quantum fluctuations can lead to an effective interaction, typically attractive, between separated bodies. A typical example is the London interaction due to quantum fluctuations, which leads to an attractive r^{-6} interaction between separated atoms or molecules.[12] There are also Keesom interactions, important in colloidal physics, which arise from thermal fluctuations in the dipole moment of a colloidal particle.[13] These interactions also die off as r^{-6}.

Because one-dimensional structures experience such large thermal fluctuations, the macroscopic coupling between these fluctuations for different lines leads to an interaction comparable in strength to direct interactions.

Parallel lines of electric dipoles offer an interesting example of this effect. For perfectly ordered lines, there is a weak interaction between lines arising from the periodic dipole moment along each line. This interaction dies off exponentially with distance, the scale of the decay being given by

the intraline dipole spacing. However, the lines will never be perfectly ordered, but will fluctuate thermally. When they so fluctuate, they develop local concentrations of dipole moments. The result is that the mean square electric field dies off as a power of distance ρ from a chain,

$$\langle E^2 \rangle \sim k_B T \frac{a}{\rho^4} \tag{1.2}$$

for dipolar particles of radius a.

This leads to a power-law induced force between the lines, arising from the coupling between these dipole fluctuations in different lines. We find that the effective free energy $F(l)$ of interaction between two fluctuating dipole chains separated by a distance l is

$$F(l) \sim -k_B T \frac{a^4 L}{l^5} \tag{1.3}$$

for chains of length L. This force is comparable in order of magnitude to but larger than the Keesom force arising from thermal fluctuations in the dipole moments of the individual particles.

For vortex lines in superconductors, we have a different situation. Often, we are interested in the regime in which the distance between vortex lines is of the same order of magnitude as the London penetration depth λ, which determines the range of the interaction between lines. In this case, the interaction between lines arising from fluctuations is strongly repulsive. The interaction is repulsive because the field of a neighboring vortex line reduces the phase space for fluctuations of a line, and thus reduces the entropy of that line. The result is a contribution to the free energy of lines separated by a distance l,

$$F(l) \sim k_B T \left(\frac{L}{\lambda}\right)^2 \frac{\exp(-l/\lambda)}{\log(\lambda/\xi)(l/\lambda)^{1/2}} \tag{1.4}$$

for vortices of length L. Here ξ is the superconducting coherence length.

Due to the divergence of the Landau–Peierls fluctuations, this interaction scales with L^2, while the ordinary direct interaction scales only as L. Thus, we find that, at least near the lower critical field H_{c1}, this repulsive interaction is much stronger than the direct repulsion between Abrikosov vortex lines. Of course, the above equation somewhat overstates the case. If the vortex lines are entangled, as may be the case near H_{c1}, then it will be more appropriate to use an entanglement correlation length in the place of L.[8] Nevertheless, the qualitative conclusion remains the same.

The interactions here studied should not be confused with entropic

interactions arising from steric effects. Such interactions arise between domain walls in two-dimensional systems, or between fluctuating surfaces in three dimensions.[14] In these cases, the spatial confinement of $(d-1)$-dimensional objects by one another leads to a power law entropic repulsion. In this paper, we study effects that are due entirely to long-range interactions between the lines.

To some extent, this study is an academic exercise, as in neither of the two cases above do we expect this interaction to significantly alter the qualitative physics of the system. Nevertheless, this interaction provides a dramatic instance of the power of Landau–Peierls fluctuations. It is striking, for example, that the fluctuation-induced interaction should so strongly dominate over direct interactions (by factors of up to 10^4) in superconductors.

This paper is divided into five sections and three appendices. In Section 2 we derive the fluctuation-induced interaction between dipole chains. In Section 3 we discuss the interaction between two fluctuating vortex lines. In Section 4 we extend this calculation to deal with fluctuations in the presence of a flux lattice. In Section 5 we argue that this interaction will not strongly affect the phase diagram of type II superconductors. Technical details are relegated to three appendices. In Appendix A, we derive the field of a wavy vortex line in a superconductor. In Appendix B, we calculate the interaction energy of two such lines. In Appendix C, we perform some sums relevant to the problem of the fluctuation effects in a flux lattice.

2. DIPOLE CHAINS

In both electrorheological fluids and in ferrofluids a field induces the formation of chains of colloidal particles. We will treat the electric field case; the magnetic field case is a trivial generalization. Consider a very long line (of length L) of particles, each with dipole moment $d\hat{z}$, with the particles aligned along the z axis with spacing a (see Fig. 1). We want to calculate the electric field at a point $(\boldsymbol{\rho}, z)$ outside of the line. Of course, we need an applied field in the z direction to induce the chaining in the first place.[6] We are more interested, however, in the field generated by the dipoles themselves, whose moments are regarded as fixed. This is equivalent to taking $\varepsilon - 1 \ll 1$, where ε is the dielectric constant of the particles.

We expect solutions of Laplace's equation to be exponential in directions perpendicular to an axis of periodicity; thus, we expect a field $\mathbf{E} \sim e^{-\rho/a}$. Throughout this paper, we will consistently refer to two-dimensional vectors in the x–y plane by $\boldsymbol{\rho}$, the magnitude of which is ρ. Three

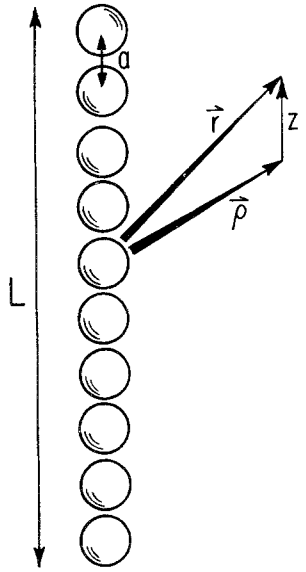

Fig. 1. A line of dipoles. The field of the line is calculated at a position $(\boldsymbol{\rho}, z)$, where $\boldsymbol{\rho}$ is a two-dimensional vector in the plane perpendicular to the line, and z is the altitude.

dimensional vectors will be indicated by **r**. Finally, we will use the two-dimensional vector $\boldsymbol{l}$ to indicate the unperturbed distance between two lines.

The z component of the electric field due to a single dipole at position $(\boldsymbol{\rho}, z) = (0, na)$ is

$$\frac{E(\rho, z)}{d} = r^{-3}\left[3\left(\frac{z-na}{r}\right)^2 - 1\right] = \rho^{-3}\frac{2\phi_n^2 - 1}{(\phi_n^2+1)^{5/2}} \equiv \rho^{-3}F(\phi_n) \quad (2.1)$$

where $r = [\rho^2 + (z-na)^2]^{1/2}$, $\phi_n = (z-na)/\rho$, and without loss of generality we can set $0 < z < a$. Then the field due to the entire chain is

$$\frac{E(\rho, z)}{d} = \rho^{-3}\sum_n F(\phi_n) = \rho^{-3}\int d\phi\left[\sum_n \delta\left(\phi - \frac{1}{\rho}(z-na)\right)\right]F(\phi) \quad (2.2)$$

Using the Poisson sum formula and the fact that $F(\phi)$ is even, we obtain

$$\frac{E(\rho, z)}{d} = \frac{1}{\rho^2 a}\sum_{m=0}^{\infty} A_m \cos\left(2\pi m\frac{z}{a}\right) \quad (2.3)$$

where

$$A_0 = \int_{-\infty}^{+\infty} d\phi\, F(\phi) = 0 \tag{2.4a}$$

$$A_m = \int_{-\infty}^{+\infty} d\phi\, F(\phi)\, e^{-2\pi i m\rho/a} \equiv I(k\rho) \tag{2.4b}$$

and $k = 2\pi m/a$. We can express $I(k\rho)$ in terms of the modified Bessel functions,[15]

$$I(k\rho) = -2(k\rho)^2 K_0(k\rho) \tag{2.5}$$

Using the asymptotic expansion of the Bessel function, we have

$$I(k\rho) \approx -(2\pi)^{1/2} (k\rho)^{3/2} e^{-k\rho} \tag{2.6}$$

for $(k\rho) \gg 1$. For $\rho \gg a$, the leading term has $m = 1$, and

$$E(\rho, z) \approx -(2\pi)^2 (d/\rho^2 a)(a/\rho)^{1/2} e^{-2\pi\rho/a} \cos(2\pi z/a) \tag{2.7}$$

The interaction energy per unit length with another column at distance l is the integral of the dipole moment density with the field,

$$U = \frac{2}{L}\int_0^L dz \left[d\sum_n \delta(z - na) \right] E(z, l) \approx -(2\pi)^2 \frac{2d^2}{l^2 a}\left(\frac{l}{a}\right)^{1/2} e^{-2\pi l/a} \tag{2.8}$$

Thus, columns separated by a distance greater than their interparticle spacing do not interact strongly.

Thermal fluctuations affect the field rather dramatically. Physically, we can make a simple argument to this effect. Suppose that there is a standing wave disturbance with wavevector k. Then, since the potential is a solution to Laplace's equation, we expect the long-distance field to go as $e^{-k\rho}\cos(kz)$. Fluctuations with wavelengths comparable to the distance l between chains will thus interact strongly with similar fluctuations in other chains.

Suppose that a chain is undergoing longitudinal fluctuations, with particle positions at $z_n = na + u_n$. The local density is $n(z) = 1/(a + u_{n+1} - u_n)$. If we replace u_n by $u(z)$, then $u_{n+1} - u_n \approx a\,\partial u/\partial z$, so at long wavelengths $n(z) \approx (1/a)(1 - \partial u/\partial z)$. The z component of the electric field due to a chain undergoing longitudinal fluctuations is

$$E(\rho, z) \equiv \hat{z} \cdot \mathbf{E}(\rho, z) = (d/\rho^2 a) \int d\phi\, F(\phi - z/\rho)\, n(\phi)$$

$$= (d/\rho^2 a) \int d\phi\, F(\phi)\, n(\phi + z/\rho) \tag{2.9}$$

The average field vanishes since $\int d\phi\, F(\phi) = 0$ and $\langle \partial u/\partial z \rangle = 0$, but higher-order expectation values do not vanish. Thus,

$$\langle E(\rho, z)\, E(\rho, z') \rangle = \frac{d^2}{\rho^4 a^2} \iint d\phi\, d\phi'\, F(\phi)\, F(\phi') \left\langle n\left(\phi + \frac{z}{\rho}\right) n\left(\phi' + \frac{z'}{\rho}\right) \right\rangle \tag{2.10}$$

The density–density correlation function is

$$\begin{aligned} \langle n(z)\, n(z') \rangle &= (1/a^2)[1 + \langle [\partial u(z)/\partial z][\partial u(z')/\partial z'] \rangle] \\ &= (1/a^2)\left[1 + (1/2\pi) \int dk\, e^{ik(z-z')} k^2 \langle |u(k)|^2 \rangle \right] \end{aligned} \tag{2.11}$$

For $\rho \gg a$ the most significant contribution to the field will come from the long-wavelength fluctuations, those for which $ka \ll 1$. For a reasonable choice of short-range interparticle interactions these phonons will have an acoustic spectrum, so equipartition gives $\langle |u(k)|^2 \rangle = A/k^2$, where $A = k_B T/mc_l^2$ and mc_l^2 is the longitudinal stiffness energy. It follows that

$$\langle E(\rho, z)\, E(\rho, z') \rangle = \frac{d^2 A}{\rho^4 a^2} \int d\phi\, F(\phi)\, F\left(\phi + \frac{z}{\rho} - \frac{z'}{\rho}\right) \tag{2.12}$$

In particular,

$$\langle E^2 \rangle = \frac{d^2 A}{\rho^4 a^2} \int d\phi\, [F(\phi)]^2 = \frac{27\pi}{128} \frac{d^2 k_B T}{\rho^4 a^2 mc_l^2} \tag{2.13}$$

We can calculate the phonon spectrum for a chain of dipolar particles, if we assume that the order of magnitude of their radius is a, and we choose a form for the short-range interparticle forces. The result is that $mc_l^2 \sim d^2 a^{-3}$. This result can also be obtained by dimensional analysis, as $d^2 a^{-3}$ gives the scale of interaction energy between the particles. Thus

$$\langle E^2 \rangle \sim \frac{k_B T a}{\rho^4} \tag{2.14}$$

The root mean square field goes as $1/\rho^2$, compared to the exponential field of a perfect chain. This field does not depend upon the dipole moment, provided that the applied field is strong enough that the particles are aggregated into chains. This is due to the fact that the stiffness of a chain increases with d, and thus the strength of the fluctuations is inversely proportional to d.

We now proceed to calculate the free energy due to interactions

between parallel dipolar chains separated by a distance l. Consider a chain that has a small deformation

$$\mathbf{u}(z) = (a_{zk}e^{ikz} + a_{z-k}e^{-ikz},\ a_{\|k}e^{ikz} + a_{\|-k}e^{-ikz},\ a_{\perp k}e^{ikz} + a_{\perp -k}e^{-ikz}) \tag{2.15}$$

where $a_{\|}$ and $a_{\perp}$ refer, respectively, to the components of the transverse fluctuation parallel and perpendicular to a line from the chain to the observation point (see Fig. 2). To first order in $\mathbf{u}$, the field due to a fluctuation $\mathbf{u}(z)$ is

$$\begin{aligned} E(\rho, z) = (2dk^3/a)[&-(ia_{zk}e^{ikz} - ia_{z-k}e^{-ikz})\,K_0(k\rho) \\ &+ (a_{\|k}e^{ikz} - a_{\|-k}e^{-ikz})\,K_1(k\rho)] \end{aligned} \tag{2.16}$$

The second-order field is proportional to the integral

$$\int_{-\infty}^{\infty} d\phi\, F(\phi)(e^{2ikz}e^{2ik\rho\phi} + e^{-2ikz}e^{-2ik\rho\phi} - 2) \tag{2.17}$$

The harmonic terms will not contribute to the second-order interaction energy, and the last term vanishes because $\int_{-\infty}^{\infty} d\phi\, F(\phi) = 0$. Thus, the second-order field does not contribute to the second-order interaction energy. Physically, this means that the energy of a fluctuation in one line is not affected at this order by the presence of a nearby straight line; the

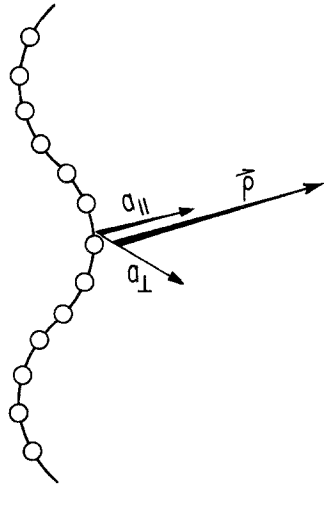

Fig. 2. Parallel and perpendicular components $a_{\|}$ and $a_{\perp}$ of a transverse fluctuation. These components are defined in terms of the vector to the observation point.

"mass" of a fluctuation is not affected by the presence of other lines. This is different from the case for superconducting vortex lines (discussed in Section 3 below), and leads to an attractive interaction between the dipole lines. At this order in **u**, the transverse perpendicular component $a_\perp$ does not contribute to the interaction energy.

The interaction energy with a chain a distance l away undergoing a fluctuation described by $(b_{zk}, b_{||k}, b_{\perp k})$ is

$$U_{\text{int},k} = (2d^2L/a^2)\, 2k^4 K_0(kl)(a_{zk}b_{z-k} + a_{z-k}b_{zk}) - (2d^2L/a^2)\, k^4[K_0(kl) + K_2(kl)](a_{||k}b_{||-k} + a_{||-k}b_{||k}) \tag{2.18}$$

The interaction energy has a sharp peak at $kl \simeq 1$, which indicates that the only modes that will interact strongly are those with $kl \sim 1$. Note that modes of different k do not interact to this order. To obtain the total interaction energy, one merely decomposes the chain fluctuations into their Fourier components and then integrates over the different wavevector contributions given by Eq. (2.18).

The elastic energy is

$$U_{\text{el},k} = (L/a)\, mc_l^2 k^2(|a_{zk}|^2 + |b_{zk}|^2) + (L/a)\, mc_t^2 k^2(|a_{||k}|^2 + |b_{||k}|^2) \tag{2.19}$$

where mc_t^2 is the transverse stiffness energy. The total energy ε_k decouples into longitudinal and transverse parts,

$$\varepsilon_{zk} = (4d^2L/a^2)\, k^4 K_0(kl)(a_{zk}b_{z-k} + a_{z-k}b_{zk}) + (L/a)\, mc_l^2 k^2(|a_{zk}|^2 + |b_{zk}|^2) \tag{2.20a}$$

and

$$\varepsilon_{||k} = (2d^2L/a^2)\, k^4[K_0(kl) + K_2(kl)](a_{||k}b_{||-k} + a_{||-k}b_{||k}) + (L/a)\, mc_l^2 k^2(|a_{||k}|^2 + |b_{||k}|^2) \tag{2.20b}$$

This contributes a factor to the partition function,

$$Z_k = \int d^3\mathbf{a}_k\, d^3\mathbf{a}_{-k}\, d^3\mathbf{b}_k\, d^3\mathbf{b}_{-k}\, e^{-\varepsilon_k/k_B T} = \frac{(2\pi k_B T)^3/(da^2k^2)^4}{[1-4(ka)^4\, K_0^2(kl)][1-(ka)^4\,(K_0(kl)+K_2(kl))^2]} \tag{2.21}$$

where $\varepsilon_k = \varepsilon_{kz} + \varepsilon_{||k}$.

The free energy for lines a distance l apart is $F(l) = -k_B TL/2\pi \int dk \log Z_k$. Since the logarithms are additive, the contributions from the longitudinal and transverse fluctuations just add.

If we expand the l-dependent part of the free energy in powers of a/l, the lowest order term is

$$F(l) = -\frac{k_B T}{2\pi} \frac{La^4}{l^5} \int_0^\infty dy\, y^4 [5K_0^2(y) + K_2^2(y) + 2K_0(y)\, K_2(y)] \tag{2.22}$$

We can numerically integrate, with the result

$$F(l) = -11.11 \frac{k_B T}{2\pi} \frac{La^4}{l^5} \tag{2.23}$$

When $2\pi l/a > \log(d^2/a^3 k_B T)$, this will be larger than the straight-line interaction energy. For typical electrorheological fluids $(k_B T a^3/d^2) \sim 10^{-4}$, so when $l/a > 3$ the straight-line interaction energy will be small compared to the fluctuation coupling energy.[6] The long-time relaxation of these fluids should be governed by this interaction.

Of course, there is another interaction, the normal van der Waals interaction between the particles. For colloidal particles, the dominant van der Waals-type interaction is the Keesom interaction, which arises from the coupling between thermal fluctuations in the dipole moment of the individual particles. The interaction energy between lines is $U_{vdW} = -A(a/2)^4 L/l^5$, where A is the Hamaker constant. For electrorheological fluids, A varies over about $(0.2\text{–}2.5)\, k_B T$, so that this coupling is an order of magnitude smaller than the fluctuation coupling discussed here.[6] Thus, the effect of this fluctuation coupling should be seen in an increase in the Hamaker constant.

3. FLUX LINES IN ISOTROPIC SUPERCONDUCTORS

Flux lines play an important role in the behavior of type II superconductors above the lower critical field H_{c1}. Within a small core of radius ξ the superconducting order parameter goes to zero, while the magnetic field extends over a radius λ, the penetration depth. Outside of the core the field $\mathbf{h}$ is governed by the London equation,

$$\left(\Delta - \frac{1}{\lambda^2}\right) \mathbf{h} = -\frac{1}{\lambda^2} \boldsymbol{\Omega}(\mathbf{r}) \tag{3.1}$$

where $\boldsymbol{\Omega}(\mathbf{r}) = \Phi_0 \hat{t} \int dz\, \delta(\mathbf{r} - \mathbf{r}_0(z))$. Here Φ_0 is the superconducting flux quantum, $\hat{t}$ is the unit tangent vector to the vortex line, and $\mathbf{r}_0(z)$ is the

position of the vortex line, which we parametrize by z. The energy of a flux line is

$$\varepsilon = \frac{\Phi_0}{8\pi} \int d^3r\, \mathbf{h} \cdot \hat{t}\delta^2(\boldsymbol{\rho} - \boldsymbol{\rho}_0(z)) \tag{3.2}$$

where we evaluate the field $\mathbf{h}$ at the surface of the region in which the superconducting order parameter goes to zero, i.e., at a radius ξ from the center of the flux line.[1]

The Green's function for the London equation is $G(\mathbf{r}-\mathbf{r}') = [\exp(-|\mathbf{r}-\mathbf{r}'|)]/4\pi\,|\mathbf{r}-\mathbf{r}'|$. For a straight flux line at the origin, $\hat{t} = \hat{z}$ and $\boldsymbol{\rho}_0(z) = 0$. The integral is then elementary, and the field is $\mathbf{h}(\mathbf{r}) = \hat{z}(\Phi_0/2\pi\lambda^2)$ $K_0(\rho/\lambda)$ in cylindrical coordinates (ρ, ϕ, z).

For $\rho/\lambda \gg 1$ this falls off exponentially in ρ/λ. The energy of the line is

$$L\varepsilon_0 = (\Phi_0/8\pi) \int_0^L dz\, \hat{t} \cdot \mathbf{h}(\xi) = L(\Phi_0/4\pi\lambda)^2 \log(\lambda/\xi)$$

where L is the length of the flux line.

Now consider a deformed flux line for which the position as a function of z is given by the two-dimensional vector $\boldsymbol{\rho}_0(z) = \mathbf{a}_k e^{ikz} + \mathbf{a}_{-k} e^{-ikz}$. We can obtain the field by integrating the Green's function with the source term $\hat{t}(z) \int dz\, \delta(\boldsymbol{\rho}' - \boldsymbol{\rho}_0(z))$. To obtain the energy to second order, we need h_z to second order, but we need the field perpendicular to the unperturbed position of the line $\mathbf{h}_\perp$, only to first order (since $\mathbf{h}_\perp$ will be multiplied by first-order terms in $\hat{t}_\perp$ in the expression for the energy). We will quote the result, leaving the details of the calculation to Appendix A:

$$\mathbf{h}_\perp = \frac{\Phi_0}{2\pi\lambda^2} (ike^{ikz}\mathbf{a}_k - ike^{-ikz}\mathbf{a}_{-k})\, K_0(\rho(k^2 + \lambda^{-2})^{1/2}) \tag{3.3}$$

and for the zeroth-order, first-order, and second-order contributions to h_z (respectively h_z^0, h_z^1, and h_z^2)

$$h_z^0 = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{\rho}{\lambda}\right) \tag{3.4a}$$

as mentioned above,

$$h_z^1 = -\frac{\Phi_0}{2\pi\lambda^2} (k^2 + \lambda^{-2})^{1/2}\, K_1(\rho(k^2 + \lambda^{-2})^{1/2})\, \hat{\rho} \cdot (\mathbf{a}_k e^{ikz} + \mathbf{a}_{-k} e^{-ikz}) \tag{3.4b}$$

and

$$h_z^2 = \frac{\Phi_0}{2\pi\lambda^2}\left\{ - |\mathbf{a}_k|^2 k^2 K_0\left(\frac{\rho}{\lambda}\right)\right.$$

$$+ \frac{1}{2}[(2k)^2 + \lambda^{-2} + k^2](\mathbf{a}_k \cdot \mathbf{a}_k e^{2ikz} + \mathbf{a}_{-k} \cdot \mathbf{a}_{-k} e^{-2ikz})$$

$$\times K_0(\rho[(2k)^2 + \lambda^{-2}]^{1/2})$$

$$+ \left\{\left[(\mathbf{a}_k \cdot \hat{\rho})^2 - \frac{\mathbf{a}_k \cdot \mathbf{a}_k}{2}\right] e^{2ikz} + \left[(\mathbf{a}_{-k} \cdot \hat{\rho})^2 - \frac{\mathbf{a}_{-k} \cdot \mathbf{a}_{-k}}{2}\right] e^{-2ikz}\right\}$$

$$\times K_2(\rho[(2k)^2 + \lambda^{-2}]^{1/2})\,[(2k)^2 + \lambda^{-2}]$$

$$+ \lambda^{-2}\left[(\mathbf{a}_k \cdot \hat{\rho})(\mathbf{a}_{-k} \cdot \hat{\rho}) - \frac{1}{2}\mathbf{a}_k \cdot \mathbf{a}_{-k}\right] K_2\left(\frac{\rho}{\lambda}\right)$$

$$\left. + \frac{1}{2}\lambda^{-2}\mathbf{a}_k \cdot \mathbf{a}_{-k} K_0\left(\frac{\rho}{\lambda}\right)\right\} \tag{3.4c}$$

We see the effects of screening here—the field of a fluctuation falls off faster than the straight-line field, unlike the field due to dipole lines in a neutral fluid. One way of checking this expression is to take the limit $k \to 0$, in which case the above expressions should be related to derivatives of $K_0(\rho/\lambda)$.

Now we want to calculate the interaction energy between modes in two lines. We expect that for long-wavelength deformations ($k \ll \lambda^{-1}$) the single-line energy of a mode should come from the lengthening of the line,

$$U_{\mathrm{el},k} = (\Phi_0/4\pi\lambda)^2 \log(\lambda/\xi) \int_0^L dz\, (1 + |d\boldsymbol{\rho}_0/dz|^2)^{1/2} \approx L\varepsilon_0(1 + k^2\,|\mathbf{a}_k|^2/2) \tag{3.5}$$

Now consider two lines, one at $\boldsymbol{\rho}_a(z) = \mathbf{a}_k e^{ikz} + \mathbf{a}_{-k} e^{-ikz}$ and one at $\boldsymbol{\rho}_b(z) = \boldsymbol{l} + \mathbf{b}_k e^{ikz} + \mathbf{b}_{-k} e^{-ikz}$. The interaction energy is

$$U_{\mathrm{int},k} = (\Phi_0/8\pi) \int_0^L dz\, \mathbf{h}_b[\boldsymbol{\rho}_a(z)] \cdot \hat{t}_a(z) + (\Phi_0/8\pi) \int_0^L dz\, \mathbf{h}_a[\boldsymbol{\rho}_b(z)] \cdot \hat{t}_b(z)$$

Just as for the dipolar chains, the total energy may be obtained by summing the different wavevector contributions. Also, the energy asso-

ciated with fluctuations in the line position normal to the vector between the two lines will, in general, be different from the fluctuations in the direction parallel to the vector between the two lines (see Fig. 3). Thus we write $\mathbf{a}_k = a_{k||}\hat{l} + a_{k\perp}(\hat{z} \times \hat{l})$. We can similiarly decompose $\mathbf{b}_k$. We then write $\varepsilon_{||}$ for the energy of the parallel fluctuations and $\varepsilon_\perp$ for the energy of the perpendicular fluctuations. (To this order there are no cross terms.) Appendix B gives the details of this calculation, with the result

$$\begin{aligned}\varepsilon_{||k} = \frac{L\Phi_0^2}{16\pi^2\lambda^2}\Bigg(&\left\{\log\left(\frac{\lambda}{\xi}\right)\frac{k^2}{2} - K_0\left(\frac{l}{\lambda}\right)2k^2 + \frac{1}{\lambda^2}\left[K_2\left(\frac{l}{\lambda}\right) + K_0\left(\frac{l}{\lambda}\right)\right]\right\} \\ &\times (|a_{||}|^2 + |b_{||}|^2) \\ &+ [(k^2 - \lambda^{-2})\,K_0(l(k^2+\lambda^{-2})^{1/2}) - (k^2+\lambda^{-2})\,K_2(l(k^2+\lambda^{-2})^{1/2})] \\ &\times (a_{||k}b_{||-k} + a_{||-k}b_{||k})\Bigg) \\ \equiv \frac{L\Phi_0^2}{16\pi^2\lambda^2}&[f_{||}(k,l)(|a_{||}|^2 + |b_{||}|^2) + g_{||}(k,l)(a_{||k}b_{||-k} + a_{||-k}b_{||k})]\end{aligned} \tag{3.6a}$$

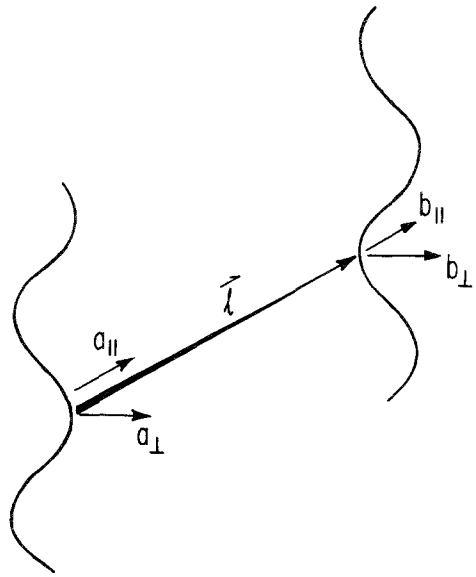

Fig. 3. Parallel and perpendicular fluctuations of two lines. The different polarizations have different energies.

and

$$\varepsilon_{\perp k} = \frac{L\Phi_0^2}{16\pi^2\lambda^2}\left(\left\{\log\left(\frac{\lambda}{\xi}\right)\frac{k^2}{2} - K_0\left(\frac{l}{\lambda}\right)2k^2 + \frac{1}{\lambda^2}\left[K_0\left(\frac{l}{\lambda}\right) - K_2\left(\frac{l}{\lambda}\right)\right]\right\}\right.$$
$$\times (|a_\perp|^2 + |b_\perp|^2) + [(k^2 - \lambda^{-2})\,K_0(l(k^2+\lambda^{-2})^{1/2})$$
$$\left. + (k^2+\lambda^{-2})\,K_2(l(k^2+\lambda^{-2})^{1/2})](a_{\perp k}b_{\perp -k} + a_{\perp -k}b_{\perp k})\right)$$
$$\equiv \frac{L\Phi_0^2}{16\pi^2\lambda^2}[f_\perp(k,l)(|a_\perp|^2 + |b_\perp|^2) + g_\perp(k,l)(a_{\perp k}b_{\perp -k} + a_{\perp -k}b_{\perp k})] \tag{3.6b}$$

We have implicitly defined the coefficients $f_{||}$, $f_\perp$, $g_{||}$, and $g_\perp$. We have also included the self-energy of the lines. Note the existence of "mass terms" in the interaction energy, which survive in the limit $k \to 0$. If $l/\lambda \gg 1$, the Bessel functions will be very small, so the l dependence of the free energy will come from the terms in the partition function which are lowest order in $K(l/\lambda)$. These terms do not go to zero as $k \to 0$; we call them mass terms.

For small enough l, the perpendicular mass term will become negative. If, however, the lines are far enough apart that the energy is still positive at the infrared cutoff, we can obtain the partition function by direct integration.

The partition function is then

$$Z_k = \int d^2a_{||k}\, d^2b_{||k}\, d^2a_{\perp k}\, d^2b_{\perp k} \exp\left[-\frac{(\varepsilon_{||k} + \varepsilon_{\perp k})}{k_B T}\right]$$
$$= \frac{(2\pi k_B T)^4}{L\Phi_0^2/16\pi^2\lambda^2}[(f_{||}^2 - g_{||}^2)(f_\perp^2 - g_\perp^2)]^{-1} \tag{3.7}$$

The total partition function is $Z = \prod_k Z_k$, and the free energy is

$$F = -k_B T \log Z = -k_B T(L/2\pi)\int dk \log Z_k \tag{3.8}$$

We are interested in the l dependence of the free energy. We can isolate the l-dependent terms as

$$F(l) = \frac{k_B TL}{2\pi}\int dk \log[(f_{||}^2 - g_{||}^2)(f_\perp^2 - g_\perp^2)] \tag{3.9}$$

Now, if the lines are far enough apart, we can retain only the lowest

order terms in the Bessel functions. Furthermore, all of the Bessel functions are equal in this limit. Thus, we obtain

$$F(l) = \frac{2k_B TL}{2\pi} \int dk \, \frac{(2\lambda^{-2} - 4k^2) \, K_0(l/\lambda)}{(k^2/2)\log(\lambda/\xi)}$$

$$= \frac{k_B TL}{2\pi} \frac{8}{\log(\lambda/\xi)} K_0\left(\frac{l}{\lambda}\right)\left[\frac{1}{\lambda^2}\left(\frac{1}{k_{\min}} - \frac{1}{k_{\max}}\right) - (k_{\max} - k_{\min})\right] \tag{3.10}$$

The ultraviolet cutoff is $k_{\max} \sim 1/\lambda$, and the infrared cutoff is $k_{\min} \sim 1/L$. Since $L \gg \lambda$, the first term will dominate, and

$$F(l) \approx \left(\frac{L}{\lambda}\right)^2 \frac{k_B T}{2\pi} \frac{8}{\log(\lambda/\xi)} K_0\left(\frac{l}{\lambda}\right) \tag{3.11}$$

If we take typical values for the parameters, then this free energy is $\sim 10^4$ times the direct interaction energy (see Section 5 below).[1]

These calculations have assumed that $l \gg \lambda$. We cannot generalize to $l \leqslant \lambda$, although one might expect the interaction to grow stronger as the lines get closer. This coupling is strongly nonlocal in z, and will not be seen in theories in which the problem of the behavior of vortex lines is mapped onto a gas of interacting bosons with equal-time interactions.

4. FLUX LINE LATTICE

Over most of the H–T phase diagram one expects the flux lines to be arranged in an Abrikosov lattice. We now examine the effects of the fluctuation-induced interaction on the free energy of the lattice. Consider a triangular lattice with spacing l. The term in the two-line energy which is proportional to $[(\mathbf{a}_k \cdot \hat{\rho})(\mathbf{a}_{-k} \cdot \hat{\rho}) - \frac{1}{2}|\mathbf{a}_k|^2]$ will vanish by symmetry when summed over the lattice. Because of this the mass terms never become negative, so the second-order partition function will be finite even at high densities. Thus, in principle, we should be able to determine the free energy for any line separation. To start, we sum the two-line energy over the lattice. If we then Fourier transform over the two-dimensional space perpendicular to $\hat{z}$, the Hamiltonian is diagonal, and the partition function may be easily calculated for a particular Fourier component. We then sum over two-dimensional Fourier components and k vectors parallel to $\hat{z}$ to obtain the free energy. The difficult step is the original Fourier transform of the interaction energy terms.

From Eqs. (3.6a) and (3.6b) we can write the kth energy component ε_k as

$$\frac{\varepsilon_k}{\bar{\varepsilon}} = \sum_{\mathbf{x}_i} \mathbf{a}_k(\mathbf{x}_i) \cdot \mathbf{a}_{-k}(\mathbf{x}_i) \left[\log(\lambda/\xi)(k^2/2) + \sum_{\mathbf{x}_j \neq \mathbf{x}_i} f_k(|\mathbf{x}_j|) \right]$$
$$+ \sum_{\mathbf{x}_i \neq \mathbf{x}_j} \mathbf{a}_k(\mathbf{x}_i) \cdot \mathbf{a}_{-k}(\mathbf{x}_j)\, g_k(|\mathbf{x}_i - \mathbf{x}_j|)$$
$$+ \sum_{\mathbf{x}_i \neq \mathbf{x}_j} \mathbf{a}_k(\mathbf{x}_i)\, \mathbf{a}_{-k}(\mathbf{x}_j) \cdot \mathsf{h}_k(\mathbf{x}_i - \mathbf{x}_j) \tag{4.1}$$

where the sum is over 2D triangular lattice positions $\mathbf{x}_i$, and following the notation of Section 3, $\mathbf{a}_k(\mathbf{x}_i)$ is the Fourier transform in the z direction of the position of the ith flux line. Here, $\bar{\varepsilon} = (L/2)(\Phi_0/4\pi\lambda)^2 = (L/2)\,\varepsilon_0$ and

$$f_k(|\mathbf{x}_i|) = (\lambda^{-2} - 4k^2)\, K_0(|\mathbf{x}_i|/\lambda_k) \tag{4.2a}$$

$$g_k(|\mathbf{x}_i - \mathbf{x}_j|) = (k^2 - \lambda^{-2})\, K_0(|\mathbf{x}_i - \mathbf{x}_j|/\lambda_k) + (k^2 + \lambda^{-2})\, K_2(|\mathbf{x}_i - \mathbf{x}_j|/\lambda_k) \tag{4.2b}$$

$$\mathsf{h}_k(\mathbf{x}_i - \mathbf{x}_j) = (k^2 + \lambda^{-2}) \frac{(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2} K_2\left(\frac{|\mathbf{x}_i - \mathbf{x}_j|}{\lambda_k}\right) \tag{4.2c}$$

where $\lambda_k = 1/(k^2 + \lambda^{-2})^{1/2}$.

We now Fourier transform the energy in the 2D plane normal to $\hat{z}$:

$$\frac{\varepsilon_k}{\bar{\varepsilon}} = \sum_{\mathbf{q}} \mathbf{a}_{k\mathbf{q}} \cdot \mathbf{a}_{-k-\mathbf{q}} \left[\log\left(\frac{\lambda}{\xi}\right)\left(\frac{k^2}{2}\right) + f_{k0} + g_{k\mathbf{q}} \right]$$
$$+ \sum_{lm} \sum_{\mathbf{q}} a_{k\mathbf{q}l} a_{-k-\mathbf{q}m} h_{k\mathbf{q}lm} \tag{4.3}$$

In this equation, k is a one-dimensional wave vector corresponding to the $\hat{z}$ direction, $\mathbf{q}$ is a two-dimensional wave vector in the plane of the lattice, and l, m are Cartesian indices. Appendix C gives the derivation of this equation.

The partition function is then $Z_k = \prod_{\mathbf{q}} Z_{k\mathbf{q}}$, where

$$Z_{k\mathbf{q}} = \int d^2a_{k\mathbf{q}}\, d^2a_{-k-\mathbf{q}} \exp - \left\{ \frac{\bar{\varepsilon}}{k_B T} \times \sum_{lm} a_{k\mathbf{q}l} a_{-k-\mathbf{q}m} \left[\log\left(\frac{\lambda}{\xi}\right)\left(\frac{k^2}{2}\right) \delta_{lm} + f_{k0}\delta_{lm} + g_{k\mathbf{q}}\delta_{lm} + h_{k\mathbf{q}lm} \right] \right\}$$
$$= \left(\frac{2\pi k_B T}{\bar{\varepsilon}}\right)^2 \left\{ \left[\log\left(\frac{\lambda}{\xi}\right)\left(\frac{k^2}{2}\right) + f_{k0} + g_{k\mathbf{q}} + h_{k\mathbf{q}xx} \right] \times \left[\log\left(\frac{\lambda}{\xi}\right)\left(\frac{k^2}{2}\right) + f_{k0} + g_{k\mathbf{q}} + h_{k\mathbf{q}yy} \right] - h_{k\mathbf{q}xy}^2 \right\}^{-1} \tag{4.4}$$

The free energy is $F = -k_B T \log Z = -k_B T \sum_{k\mathbf{q}} \log Z_{k\mathbf{q}}$. The sum over $\mathbf{q}$ can be converted into an integral over the first Brillouin zone of the lattice. If l is the lattice spacing,

$$F = -k_B T (L/2\pi)(l/2\pi)^2 \int dk \int d^2q \log Z_{k\mathbf{q}} \tag{4.5}$$

In general, we cannot evaluate the necessary sums for arbitrary l, but in the dilute limit, where nearest neighbor interactions dominate, the sums are easy (see Appendix C). In this limit f, g, and h are all small compared to the line lengthening energy $\log(\lambda/\xi)k^2/2$, so we can expand the integrand. Retaining only the lowest order l-dependent terms, we get

$$F(l) = k_B T (L/2\pi)(l/2\pi)^2 \int dk \int d^2q \, (2f_{k0} + 2g_{k\mathbf{q}} + h_{k\mathbf{q}xx} + h_{k\mathbf{q}yy}) \tag{4.6}$$

Substituting in the expressions from Appendix C, Eqs. (C.4), and performing the integrals, we obtain the free energy

$$F(l) = k_B T \left(\frac{L}{\lambda}\right)^2 \frac{12}{\pi^{3/2} \log(\lambda/\xi)} \frac{\exp(-l/\lambda)}{(l/\lambda)^{1/2}} \tag{4.7}$$

in the limit where we retain only the infrared terms in the k integration. This is valid for $L \gg \lambda$.

5. CONCLUSIONS

For magnetic flux lines, the principal result is that the repulsion between flux lines is considerably enhanced by entropic effects, at least near H_{c1}.

Nelson and various other authors have recently advanced the claim that fluctuations can melt an Abrikosov flux lattice, either near H_{c2} or in a thin reentrant "sliver" of the phase diagram immediately above H_{c1}, which separates the Abrikosov flux lattice phase from the Meissner flux-expelling phase below H_{c1}. In this regime, the flux lines interact weakly due to their separation, and the Landau–Peierls fluctuations lead the lines to wander considerably. One can define an entanglement correlation length ξ_z, which for an isotropic superconductor is given by

$$\xi_z = \frac{\varepsilon_0}{k_B T n} \tag{5.1}$$

where n is the areal density of vortex lines and ε_0 is the line tension, given by Eq. (3.5). Thus, we expect that our assumption that the fluctuations of line position will be small compared to the distance between lines will fail above a length scale of ξ_z. Thus, we estimate that the effective fluctuation-induced free energy of interaction per unit length between lines separated by a distance l will be

$$F_f(l) \approx \frac{\xi_z}{\lambda^2} \frac{12 k_B T}{\pi^{3/2} \log(\lambda/\xi)} \frac{\exp(-l/\lambda)}{(l/\lambda)^{1/2}} \tag{5.2}$$

It is important not to confuse the entanglement length ξ_z with the superconducting coherence length ξ. We should compare this fluctuation-induced free energy with the direct interaction energy,

$$F_0(l) = 2 \left(\frac{\Phi_0}{4\pi\lambda} \right)^2 K_0 \left(\frac{l}{\lambda} \right) \tag{5.3}$$

The ratio between these two energies is of the order of

$$\frac{F_f}{F_0} \sim \frac{\xi_z k_B T}{\Phi_0^2 \log(\lambda/\xi)} \tag{5.4}$$

Because $\xi_z \propto n^{-1}$, this ratio will always go to infinity as $H \to H_{c1}$. Near H_{c1} we expect to see an interaction contribution to the free energy per unit volume F_v of a flux lattice of density n of

$$F_v \approx \frac{\varepsilon_0}{\lambda^{3/2}} \frac{12 n^{1/4}}{\pi^{3/2} \log(\lambda/\xi)} \exp\left(\frac{-1}{\lambda \sqrt{n}} \right) \tag{5.5}$$

However, we do not expect that this effect will significantly alter the qualitative physics of the superconductors near H_{c1}. This is because the entropic fluctuations of the individual lines dominate in this limit, leading to an "asymptotically free" theory in which the interaction between lines is negligible in the long-wavelength limit.[8] Near H_{c1}, the fluctuation-induced repulsion is suppressed by the exponential factor in Eq. (5.2), just as is the direct repulsion.

APPENDIX A. FIELD OF A WAVY LINE IN A SUPERCONDUCTOR

The magnetic field in a superconductor satisfies the London equation,

$$\left(\Delta - \frac{1}{\lambda^2} \right) \mathbf{h} = -\frac{1}{\lambda^2} \boldsymbol{\Omega}(\mathbf{r}) \tag{A.1}$$

where the source term $\mathbf{\Omega}(\mathbf{r})$ gives the source density for magnetic flux lines at $\mathbf{r}$. In this Appendix we will calculate the field around a single vortex line with a simple harmonic modulation. The field for a lattice of lines with more general fluctuations may then be obtained by superposition.

We will consider a line arranged on average at a position $\boldsymbol{\rho}=0$ and oriented parallel to the z axis. All delta functions of $\boldsymbol{\rho}$ will be two-dimensional delta functions. We write $\mathbf{\Omega}(\mathbf{r})=\Phi_0\hat{t}(z)\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0(z))$ and $\boldsymbol{\rho}_0(z)=\mathbf{a}_k e^{ikz}+\mathbf{a}_{-k}e^{-ikz}$ for a simple harmonic distortion of the vortex line. As in the above, $\mathbf{a}_k$ is a two-dimensional vector.

The tangent vector will be the unit vector in the direction given by

$$\mathbf{t}=\frac{d}{dz}(\mathbf{a}_k e^{ikz}+\mathbf{a}_{-k}e^{-ikz}, z)=(ik\mathbf{a}_k e^{ikz}-ik\mathbf{a}_{-k}e^{-ikz}, 1) \tag{A.2}$$

Then the unit vector $\hat{t}=\mathbf{t}/\sqrt{t^2}$, so

$$\begin{aligned}\hat{t}=\hat{z}[1-(k^2/2)(2\,|\mathbf{a}_k|^2-\mathbf{a}_k^2 e^{2ikz}-\mathbf{a}_{-k}^2 e^{-2ikz})]\\ +ik(\mathbf{a}_k e^{ikz}-\mathbf{a}_{-k}e^{-ikz})+O(a_k^3)\end{aligned} \tag{A.3}$$

Using the Green's function for the Helmholtz equation, we obtain the field as

$$\begin{aligned}\mathbf{h}(z,\boldsymbol{\rho})=\frac{\Phi_0}{4\pi\lambda^2}\iint dz'\,d^2\rho'\,\hat{t}(z')\\ \times\frac{\exp\{-[(z-z')^2+|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2]^{1/2}/\lambda\}}{[(z-z')^2+|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2]^{1/2}}\,\delta(\boldsymbol{\rho}'-\boldsymbol{\rho}_0(z'))\end{aligned} \tag{A.4}$$

We only need $\mathbf{h}_\perp$ to first order in $\mathbf{a}_k$. We calculate this by integrating over the delta function and then expanding. We obtain

$$\begin{aligned}\mathbf{h}_\perp=\frac{\Phi_0}{4\pi\lambda^2}\int dz'\,[ik\mathbf{a}_k\exp(ikz')-ik\mathbf{a}_{-k}\exp(-ikz')]\\ \times\frac{\exp\{-[(z-z')^2+|\boldsymbol{\rho}|^2]^{1/2}/\lambda\}}{[(z-z')^2+|\boldsymbol{\rho}|^2]^{1/2}}\\ =\frac{\Phi_0}{2\pi\lambda^2}[ik\mathbf{a}_k\exp(ikz)-ik\mathbf{a}_{-k}\exp(-ikz)]\,K_0(\rho(k^2+\lambda^{-2})^{1/2})\end{aligned} \tag{A.5}$$

where we have used the identity

$$\int_{-\infty}^{+\infty}dz\,\frac{\exp[ikz-(z^2+\rho^2)^{1/2}/\lambda]}{(z^2+\rho^2)^{1/2}}=2K_0(\rho(k^2+\lambda^{-2})^{1/2}) \tag{A.6}$$

K_0 is the zeroth-order modified Bessel function.[15]

We need to know the z component of the field to second order in $\mathbf{a}_k$, which requires a slightly more complicated calculation. After performing the integration over the delta function, we have

$$h_z(z, \boldsymbol{\rho}) = \frac{\Phi_0}{4\pi\lambda^2} \int dz' \left\{ 1 - \frac{k^2}{2} \left[2 |\mathbf{a}_k|^2 - \mathbf{a}_k^2 \exp(2ikz') - \mathbf{a}_{-k}^2 \exp(-2ikz') \right] \right\} \times \frac{\exp\{ -[(z-z')^2 + |\boldsymbol{\rho} - \boldsymbol{\rho}_0(z')|^2]^{1/2}/\lambda \}}{[(z-z')^2 + |\boldsymbol{\rho} - \boldsymbol{\rho}_0(z')|^2]^{1/2}} \tag{A.7}$$

We can expand the integrand in powers of ρ_0/r, where $r^2 = (z-z')^2 + \rho^2$. To second order in ρ_0/r,

$$\frac{\exp\{ -[(z-z')^2 + |\boldsymbol{\rho} - \boldsymbol{\rho}_0(z')|^2]^{1/2}/\lambda \}}{[(z-z')^2 + |\boldsymbol{\rho} - \boldsymbol{\rho}_0(z')|^2]^{1/2}} = \frac{\exp(-r/\lambda)}{r} \left[1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho}_0 \left(\frac{1}{r^2} + \frac{1}{r\lambda} \right) - \boldsymbol{\rho}_0^2 \left(\frac{1}{2r^2} + \frac{1}{2r\lambda} \right) + (\boldsymbol{\rho} \cdot \boldsymbol{\rho}_0)^2 \left(\frac{1}{2r^2\lambda^2} + \frac{1}{2r^3\lambda} + \frac{3}{2r^4} \right) \right] \tag{A.8}$$

Now we need to evaluate several integrals. The first is identical to Eq. (A.6). Taking $d/d\rho$ of both sides of Eq. (A.6), we obtain

$$\int_{-\infty}^{+\infty} dz \frac{\exp[ikz - (z^2+\rho^2)^{1/2}/\lambda]}{z^2+\rho^2} \left[\frac{1}{\lambda} + \frac{1}{(z^2+\rho^2)^{1/2}} \right] = \frac{2}{\rho} (k^2 + \lambda^{-2})^{1/2} K_1(\rho(k^2+\lambda^{-2})^{1/2}) \tag{A.9}$$

Here K_1 is again a Bessel function. If we now take the derivative again, and use the Bessel function recursion relations, we obtain the integral

$$\int_{-\infty}^{+\infty} dz \frac{\exp[ikz - (z^2+\rho^2)^{1/2}/\lambda]}{(z^2+\rho^2)^{3/2}} \left(\frac{1}{\lambda^2} + \frac{1}{r\lambda} + \frac{3}{r^2} \right) = \frac{k^2+\rho^2}{2\rho^2} K_2(\rho(k^2+\lambda^{-2})^{1/2}) \tag{A.10}$$

Putting together Eqs. (A.6)–(A.10), and using the fact that $K_1(x) = (x/2)[K_2(x) - K_0(x)]$, we obtain Eq. (3.4) for h_z. This solution has the proper form, being a sum of solutions of the homogeneous Helmholtz equation, and it has the proper $k = 0$ limit.

This calculation becomes much more difficult for the case of an anisotropic superconductor, due to the fact that the equation satisfied by the magnetic field is no longer a simple Helmholtz equation.

APPENDIX B. INTERACTION ENERGY OF TWO WAVY LINES IN A SUPERCONDUCTOR

In this appendix we will calculate the interaction energy of two fluctuating lines in a superconductor. Without loss of generality, we can regard these lines as fluctuating harmonically, with a wave vector k (as in Appendix A above). As in Section 3, let us label these lines a and b, with the position of line a being given by $\boldsymbol{\rho}_a(z) = \mathbf{a}_k \exp(ikz) + \mathbf{a}_{-k} \exp(-ikz)$, and the position of line b given by $\boldsymbol{\rho}_b(z) = \boldsymbol{l} + \mathbf{b}_k \exp(ikz) + \mathbf{b}_{-k} \exp(-ikz)$. The interaction energy is

$$U_{\text{int}} = (\Phi_0/8\pi) \iint dz\, d^2\rho\, \mathbf{h}_a \cdot \hat{t}_b \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_b(z)) + (\Phi_0/8\pi) \iint dz\, d^2\rho\, \mathbf{h}_b \cdot \hat{t}_a \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_a(z)) \tag{B.1}$$

As in the above, delta functions of $\boldsymbol{\rho}$ are two-dimensional delta functions. Let us focus our attention on the first of these integrals. Only those terms in Eq. (B.1) in which there is initially no dependence on z will remain after integrating over z, due to the harmonic dependence of the fluctuations. Using subscripts to indicate order in the fluctuations, we can write immediately three obvious terms: the nonharmonic parts of $\mathbf{h}_0^a \cdot \hat{t}_2^b$, $\mathbf{h}_2^a \cdot \hat{t}_0^b$, and $\mathbf{h}_1^a \cdot \hat{t}_1^b$. In addition, there are two terms arising from the harmonic dependence of the argument of the delta function.

The first three terms are, respectively,

$$U_1 = -\frac{L\Phi_0^2}{16\pi^2\lambda^2} k^2 |\mathbf{a}_k|^2 K_0\left(\frac{l}{\lambda}\right) \tag{B.2}$$

$$U_2 = \frac{L\Phi_0^2}{16\pi^2\lambda^2}\left(-k^2 |\mathbf{a}_k|^2 K_0\left(\frac{l}{\lambda}\right) + \frac{1}{\lambda^2}\left\{\left[(\mathbf{a}_k \cdot \hat{l})(\mathbf{a}_{-k} \cdot \hat{l}) - \frac{1}{2}|\mathbf{a}_k|^2\right] K_2\left(\frac{l}{\lambda}\right) + \frac{1}{2}|\mathbf{a}_k|^2 K_0\left(\frac{l}{\lambda}\right)\right\}\right) \tag{B.3}$$

$$U_3 = \frac{L\Phi_0^2}{16\pi^2\lambda^2} k^2 (\mathbf{a}_k \cdot \mathbf{b}_{-k} + \mathbf{a}_{-k} \cdot \mathbf{b}_k)\, K_0(l(k^2 + \lambda^{-2})^{1/2}) \tag{B.4}$$

The next term comes from evaluating $\mathbf{h}_1^a \cdot \hat{t}_0^b$ over the first-order expansion of the delta function,

$$U_4 = \frac{\Phi_0^2}{16\pi^2\lambda^2} \int dz\, d^2\rho\, (\hat{\rho} \cdot \mathbf{a}_k e^{ikz} + \hat{\rho} \cdot \mathbf{a}_{-k} e^{-ikz}) \times (k^2 + \lambda^{-2})^{1/2} K_1((k^2 + \lambda^{-2})^{1/2} |\boldsymbol{\rho}|) \times [\delta(\boldsymbol{\rho} - \boldsymbol{l} - (\mathbf{b}_k e^{ikz} + \mathbf{b}_{-k} e^{-ikz})) - \delta(\boldsymbol{\rho} - \boldsymbol{l})] \tag{B.5}$$

We can integrate over the delta function, and then expand the integrand to obtain

$$U_4 = \frac{\Phi_0^2}{16\pi^2\lambda^2} \int dz\, (a_{ki} e^{ikz} + a_{-ki} e^{-ikz})(b_{kj} e^{ikz} + b_{-kj} e^{-ikz}) \frac{k^2 + \lambda^{-2}}{2} \times \left\{ \delta_{ij} (K_2(l[k^2 + \lambda^{-2}]^{1/2}) - K_0(l[k^2 + \lambda^{-2}]^{1/2})) - 2 \frac{l_i l_j}{l^2} K_2(l(k^2 + \lambda^{-2})^{1/2}) \right\} \tag{B.6}$$

where i, j are Cartesian indices. This is

$$U_4 = L \frac{\Phi_0^2}{16\pi^2\lambda^2} \frac{1}{2} (k^2 + \lambda^{-2}) \times \{(\mathbf{a}_k \cdot \mathbf{b}_{-k} + \mathbf{a}_{-k} \cdot \mathbf{b}_k)[K_2(l(k^2 + \lambda^{-2})^{1/2}) - K_0(l(k^2 + \lambda^{-2})^{1/2})] - 2[(\hat{l} \cdot \mathbf{a}_k)(\hat{l} \cdot \mathbf{b}_{-k}) + (\hat{l} \cdot \mathbf{a}_{-k})(\hat{l} \cdot \mathbf{b}_k)]\, K_2(l(k^2 + \lambda^{-2})^{1/2})\} \tag{B.7}$$

The final term comes from the second-order expansion of the delta function over the zeroth-order field,

$$U_5 = \frac{\Phi_0^2}{16\pi^2\lambda^2} \int dz\, (\mathbf{b}_k e^{ikz} + \mathbf{b}_{-k} e^{-ikz})(\mathbf{b}_k e^{ikz} + \mathbf{b}_{-k} e^{-ikz}) \cdot \nabla_\perp \nabla_\perp K_0\left(\frac{l}{\lambda}\right) = L \frac{\Phi_0^2}{16\pi^2\lambda^2} \frac{1}{\lambda^2} \left\{ \left[(\mathbf{b}_k \cdot \hat{l})(\mathbf{b}_{-k} \cdot \hat{l}) - \frac{1}{2} |\mathbf{b}_k|^2 \right] K_2\left(\frac{l}{\lambda}\right) + \frac{1}{2} |\mathbf{b}_k|^2 K_0\left(\frac{l}{\lambda}\right) \right\} \tag{B.8}$$

If we sum up all these terms, add in the equivalent terms with a and b interchanged, add in the self-energy of the lines, and group into parallel and perpendicular polarizations, we obtain Eqs. (3.6a) and (3.6b).

APPENDIX C. ENERGY OF A FLUX LATTICE

We have Eqs. (4.1)–(4.2) for the interaction energy of the lattice as a sum over real space. In order to calculate the partition function, we must diagonalize this energy by expressing it as a sum over reciprocal lattice sites.

Define $\mathbf{a}_{k\mathbf{q}}$ and $f_{k\mathbf{q}}$ by

$$\mathbf{a}_k(\mathbf{x}_i) = \sum_{\mathbf{q}} \mathbf{a}_{k\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}_i) \tag{C.1a}$$

$$f_k(\mathbf{x}_j) = \sum_{\mathbf{q}} f_{k\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}_j) \tag{C.1b}$$

with similar expressions for g and h. The momentum k refers to the z direction in reciprocal space; the vector $\mathbf{q}$ is a two-dimensional vector in the plane of the lattice. The first term in the energy is the sum

$$\begin{aligned} S_0 &= \sum_{\mathbf{x}_i} \mathbf{a}_k(\mathbf{x}_i)\cdot\mathbf{a}_{-k}(\mathbf{x}_i)\log(\lambda/\xi)k^2/2 \\ &= \sum_{\mathbf{q}} \mathbf{a}_{k\mathbf{q}}\cdot\mathbf{a}_{-k-\mathbf{q}}\log(\lambda/\xi)k^2/2 \end{aligned} \tag{C.2a}$$

The next term in the energy is the sum

$$\begin{aligned} S_1 &= \sum_{\mathbf{x}_i} \mathbf{a}_k(\mathbf{x}_i)\cdot\mathbf{a}_{-k}(\mathbf{x}_i) \sum_{\mathbf{x}_j\neq\mathbf{x}_i} f_k(\mathbf{x}_j) \\ &= \sum_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3} \mathbf{a}_{k\mathbf{q}_1}\cdot\mathbf{a}_{-k\mathbf{q}_2} f_{k\mathbf{q}_3} \sum_{\mathbf{x}_i\neq\mathbf{x}_j} \exp[i(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{x}_i + \mathbf{q}_3\cdot\mathbf{x}_j]) \\ &= \sum_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3} \mathbf{a}_{k\mathbf{q}_1}\cdot\mathbf{a}_{-k\mathbf{q}_2} f_{k\mathbf{q}_3}\delta_{\mathbf{q}_1+\mathbf{q}_2,0}\delta_{\mathbf{q}_3,0} \\ &= \sum_{\mathbf{q}} \mathbf{a}_{k\mathbf{q}}\cdot\mathbf{a}_{-k-\mathbf{q}} f_{k0} \end{aligned} \tag{C.2b}$$

The term involving sums of h_k is

$$\begin{aligned} S_2 &= \sum_{\mathbf{x}_i\neq\mathbf{x}_j} \mathbf{a}_k(\mathbf{x}_i)\,\mathbf{a}_{-k}(\mathbf{x}_j)\cdot\mathsf{h}_k \\ &= \sum_{lm} \sum_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3} a_{k\mathbf{q}_1 l} a_{k\mathbf{q}_2 m} h_{k\mathbf{q}_3 lm} \sum_{\mathbf{x}_i\neq\mathbf{x}_j} \exp[i(\mathbf{q}_1+\mathbf{q}_3)\cdot\mathbf{x}_i + i(\mathbf{q}_2-\mathbf{q}_3)\cdot\mathbf{x}_j] \\ &= \sum_{lm} \sum_{\mathbf{q}} a_{k\mathbf{q}l} a_{-k-\mathbf{q}m} h_{k-\mathbf{q}lm} \end{aligned} \tag{C.2c}$$

Here l and m are two-dimensional Cartesian indices, and k and $\mathbf{q}$ have the same meaning as above. A similar calculation gives the term involving g_k as

$$S_3 = \sum_{\mathbf{q}} \mathbf{a}_{k\mathbf{q}} \cdot \mathbf{a}_{-k-\mathbf{q}} g_{k\mathbf{q}} \tag{C.2d}$$

If we add S_0, S_1, S_2 and S_3 together, we obtain Eq. (4.3).

Here

$$f_{k0} = \frac{1}{2\pi}(\lambda^{-2} - 4k^2) \sum_{\mathbf{x}_j \neq 0} K_0\left(\frac{x_j}{\lambda}\right) \tag{C.3a}$$

$$g_{k,\mathbf{q}} = \frac{1}{2\pi} \sum_{\mathbf{x}_j \neq 0} \exp(-i\mathbf{q} \cdot \mathbf{x}_j)\left[(k^2 - \lambda^{-2}) K_0\left(\frac{x_j}{\lambda_k}\right) + (k^2 + \lambda^{-2}) K_2\left(\frac{x_j}{\lambda_k}\right)\right] \tag{C.3b}$$

and

$$h_{k\mathbf{q}lm} = \frac{1}{2\pi} 2(k^2 + \lambda^{-2}) \sum_{\mathbf{x}_j \neq 0} \exp(-i\mathbf{q} \cdot \mathbf{x}_j) \frac{x_{jl} x_{jm}}{|\mathbf{x}_j|^2} K_2\left(\frac{x_i}{\lambda_k}\right) \tag{C.3c}$$

where, as in the above, $\lambda_k = 1/(k^2 + \lambda^{-2})^{1/2}$.

In general, we cannot evaluate these sums analytically, although we can do so in the dilute limit, where the line separation l is large compared to λ. There we retain only nearest neighbor interactions. Also, in this limit, $K_0(l/\lambda) = K_2(l/\lambda)$. Then, for a triangular lattice with the base of the triangle along the x axis,

$$f_{k0} = \frac{1}{\pi}(3\lambda^{-2} - 12k^2) K_0\left(\frac{l}{\lambda}\right) \tag{C.4a}$$

$$g_{k\mathbf{q}} = \frac{2k^2}{\pi} K_0\left(\frac{l}{\lambda_k}\right)\left[\cos(q_x l) + 2\cos\left(\frac{q_x l}{2}\right)\cos\left(\frac{\sqrt{3}\, q_y l}{2}\right)\right] \tag{C.4b}$$

$$h_{xx} = \frac{k^2 + \lambda^{-2}}{\pi} K_0\left(\frac{l}{\lambda_k}\right)\left[2\cos(q_x l) + \cos\left(\frac{q_x l}{2}\right)\cos\left(\frac{\sqrt{3}\, q_y l}{2}\right)\right] \tag{C.4c}$$

$$h_{yy} = \frac{3(k^2 + \lambda^{-2})}{\pi} K_0\left(\frac{l}{\lambda_k}\right)\cos\left(\frac{q_x l}{2}\right)\cos\left(\frac{\sqrt{3}\, q_y l}{2}\right) \tag{C.4d}$$

In this limit, h_{xy} does not contribute to the free energy.

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